The concept and role of theory in mathematics education

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"In theory there is no difference between theory and practice, but in practice there is"
(attributed to US baseball player Yogi Berra)

Introduction

The notion and term of "theory" are essential in any discipline that perceives itself as scholarly or scientific. Hence "theory" is also essential in mathematics education as a research domain, where the term is frequently used in papers, books, and – not least – in Ph.D. dissertations.

On closer inspection, however, it is not clear at all what "theory" actually means in mathematics education. Nor is it clear where the entities referred to as theories invoked in mathematics education come from, how they are developed, what foundations they have, or what roles they play in the field. It seems problematic that a key entity for the furthering of research in a field is ill-defined and has an unclear status and function in the field. Therefore, we have to pay much more attention to non-trivial issues such as the ones just mentioned. As far as I can tell, in the international debate not much attention has been paid to these issues so far, even though the word "theory" comes up more and more often in research publications. The present paper is an attempt to contribute to placing these issues on the agenda of articulate and reflective discussions in our field. It is not the intention with this paper to survey or review different specific theories put to use in mathematics education research but to offer some general considerations of an overarching nature.

The general notion of theory

For a start let us consult a few dictionaries and encyclopedias to obtain a first approximation of the general notion of theory. As regards etymology, the root of the word is Greek, *theoria,* which means consideration or speculation and is derived from the verb *theorein*: to consider. In Aristotle, "theory" refers, more specifically, to deliberations on ideas, non-empirical matters or abstract contexts without regard to action or practice (Den Store Danske Encyklopædi, 2000).

The Collins Cobuild (1999(95)) *English Dictionary* lists four different meanings of the term theory.

1. "A *theory* is a formal idea or set of ideas that is intended to **explain something** [my emphasis].",

in other words a more or less elaborate, connected and substantiated *edifice of formal concepts*, including *mechanisms for explanation*.

2. "If you have a *theory* about something, you have your **own opinion** [my emphasis] about it which you cannot prove but which you think is true.",

in other words an unsettled *hypothesis* adhered to by someone.

3. "The theory of a practical subject or skill is the set of **rules and principles** [my emphasis] that form the basis of it.",

in other words, the foundational *underpinning of a certain practice*.

4. "You use *in theory* to say that although something is supposed to be true or to happen in the way stated, **it may not in fact be true or happen that way** [my emphasis].",

which points to the *difference* between more or less established *assumptions or predictions* and hard *reality*.

Princeton's *Wordreference.com* is not too different. It presents three different meanings of the word:

1. "a *tentative* theory about the natural world; a concept that is **not yet verified** [my emphasis] but that if true would explain certain facts or phenomena.",

in other words an unsettled *hypothesis / a system of hypotheses* that may or may not hold; this more or less merges (parts of) the two first meanings in the Collins Cobuild dictionary.

2. "a *belief* that can **guide behaviour** [my emphasis]."

i.e. a conviction forming the *basis for action*, close to the third meaning in the Collins Cobuild dictionary.

Finally:

3. "a well-substantiated *explanation* of some aspect of the natural world; an organized system of accepted knowledge that applies in a variety of circumstances to **explain a specific set of phenomena** [my emphasis]."

i. e. an established and connected *edifice of general claims* covering several special cases.

On closer examination it turns out that most of these different meanings of the term "theory" are actually represented in publications within the discipline of mathematics education. We shall return to this point later in this paper.

In order for the subsequent discussion to have a clear focus it seems warranted to present a further approximation to the concept of "theory". Allow me to present my own definition of the concept:

A *theory* is a system of concepts and claims with certain properties, namely

- The theory consists of an *organised network of concepts* (including ideas, notions, distinctions, terms etc.) and *claims* about some extensive domain, or a class of domains, of objects, situations and phenomena.

- In the theory, the *concepts are linked in a connected hierarchy* (oftentimes of a logical or proto-logical nature), in which a certain set of concepts, taken to be basic, are used as building blocks in the formation of the other concepts in the hierarchy.

- In the theory, the *claims are either* basic hypotheses, assumptions, or axioms, taken as *fundamental* (i.e. not subject to discussion within the boundaries of the theory itself), or statements obtained from the fundamental claims by means of *formal or material* (by "material" we mean experiential or experimental) *derivation* (including reasoning).

In principle, for a system of concepts and claims to be called a theory, the system has to be *stable*, i.e. unchanged over a longer span of time, *coherent*, i.e. the components of the system have to be linked in a clear and non-contradictory way, and *consistent* in the sense that it is not possible to arrive at contradictory claims by means of the types of derivation permitted in the theory. In practice, however, many systems named theories do not possess all these features. This is particularly true of theories in development. Some theorists would add the requirement that all the non-fundamental claims in a theory have to be testable (or at least falsifiable) by logical or empirical confrontation with the domain(s) covered by the system. In other words, such theorists would discard what is sometimes called transcendental theories, i.e. theories in which the concepts and claims are so general and overarching that they do not apply in a straightforward way to a specific, empirically well-defined world. As numerous theories belonging to the humanities or the social sciences and employed in mathematics education are transcendental in this sense, I do not find it reasonable to exclude them from consideration here.

Theories *differ* in a multitude of respects. More specifically they differ with respect to the origin, nature and state of

- the *domain* (or class of domains) - and the kind of objects, situations and phenomena that populate it - which the theory refers to or is meant to cover;
- the *concepts* involved in the theory, and the conceptual *hierarchy* within which the concepts are organised;
- the *claims* actually or potentially made within the theory, including the fundamental claims underlying the entire theory;
- the way in which the *network* of concepts and claims of the theory is organised;
- the way in which claims are *derived and justified* within the theory;
- the degrees of *stability*, *coherence*, and *consistence* of the theory.

The variability within each of these features of a theory is sufficient to suggest that the notion of theory is not exactly a monolithic one. This observation gains even more momentum if we go on to consider the different purposes that theories may have in research. I have identified six such purposes. Some of these purposes are represented in, or are similar to, the dictionary entries quoted above. Of course, a theory can have several of these purposes at the same time.

One purpose of a theory is to provide *explanation* of some observed phenomenon occurring within a domain covered by the theory. Explaining a phenomenon by way of a theory means that the occurrence of the phenomenon can be obtained as a claim (or a consequence of a claim) in the theory. Explaining an observed phenomenon also includes specifying the conditions under which it occurred, and substantiating the fulfilment of these conditions within the theory.

Another purpose is to provide *predictions* of the (possible) occurrence of certain phenomena. Again, this means to establish the occurrence of the phenomena at issue as (possible) claims in the theory as a result of the (possible) fulfilment of the preconditions for their occurrence. It appears that there may well be a close link between explanation and prediction, since prediction, including a choice between possible scenarios, will often (but not necessarily always) rely on explanation of causes and mechanisms, and since explanation will often (but not necessarily always) give rise to predictions.

A third purpose is to provide *guidance for action or behaviour* by employing knowledge of claims, and the conditions of their validity in the theory, in order to plan and implement action or behavior so as to achieve desirable – or to avoid undesirable – outcomes.

A fourth purpose is to provide *a structured set of lenses* through which aspects or parts of the world can be approached, observed, studied, analysed or interpreted. This takes place by selecting the elements that are important for consideration in the context, while omitting others; by focusing on certain features or issues; by adopting and utilising particular perspectives; and by providing a *methodology* for the whole enterprise.

Yet another purpose is to provide *a safeguard against unscientific approaches* to a problem, an issue or a theme, uncluding, for example, haphazard and inconsistent choices with regard to terminology, research metholodology, and interpretation of results. This purpose is pursued by articulating underlying assumptions and choices and by making them explicit and subject to discussion; by situating one's research within some framework; and by declaring and describing its characteristics vis-à-vis possible alternatives.

The sixth and final purpose is to provide *protection against attacks* from skeptical or hostile colleagues in other disciplines. For instance, any mathematics education researcher has experienced criticism of our field from outside colleagues (especially in mathematics, psychology, or general education) concerning the foundation of our research and its results. And researchers in the humanities and social sciences at large have often encountered similar criticism. For quite a few of us / them the invocation of one or more theories, claimed to underpin research, may serve the purpose of counteracting such criticism.

It is now time to leave the general treatment of the concept and role of theory aside and move on to the part it plays in the field of mathematics education research.

***Theories* in mathematics education**

The questions concerning theories put to use in mathematics education research that will preoccupy us in this section are the following:

- What are they?
- Where do they come from?
- What foundations do they have?
- What are their roles?
- Are they "good enough" (in a sense that has to be specified)?
- What should we strive at in terms of theory development and use?

What are the theories put to use in mathematics education?
To begin with, four general observations pertaining to this question should be kept in mind.

Firstly, there is *no such thing* as a well-established unified "theory of mathematics education" which is supported by the majority of mathematics education researchers. On the contrary, different groups of researchers represent different schools of thought, some of which appear to be mutually incompatible if not outright contradictory. Moreover, for reasons that will be discussed in a later section of this paper it is not likely that we shall get a unified theory of mathematics education in a foreseeable future, if ever.

The scond observation is that many mathematics education researchers relate their work to explicitly invoked theories *borrowed from other fields* (or at least from authors who belong to other fields), and often do so in rather eclectic or vague ways. Only rarely are theories "home-grown" within mathematics education itself.

Thirdly, much discussion and debate in mathematics education research takes the shape of *"fights"* with and between theories. This may be potentially fruitful to the extent competing theories offer different perspectives on the same thing, whereas it is potentially futile, if not destructive, if the theories deal with different things and therefore are only competing in the superficial sense that "my topic object of study is more important than yours".

Finally, the fourth observation is that quite a few mathematics education *researchers do not explicitly invoke or employ any theory* at all in their work. Furthermore, many researchers who actually do invoke a theory in their publications do not seem to go beyond the mere invocation. In other words, some theoretical framework may be referred to in the beginning or in the end of a paper without having any presence or bearing on what happens between the beginning and the end.

Against this background, what theories are in fact being put to use by researchers in mathematics education? Let us begin by looking at theories that are, in principle, extraneous to mathematics education but which have been "imported" into the field.

First of all we find theories about the epistemology and sociology of *mathematics as a discipline* (e.g. Davis and Hersh, 1980; Ernest, 1991; von Glasersfeld, 1995; Hersh, 1997; Kitcher, 1984; Lakatos, 1976; Tymoczko, 1985), This is hardly surprising, as problems encountered in the teaching and learning of mathematics are often closely linked to the nature of mathematics as a discipline and to its subjective and objective relevance to life in society. Typically, theories about mathematics as a discipline, such as formalism, structuralism, empiricism, radical constructivism, social constructivism, semiotics, etc., relate to more general theories of knowledge or of social practice. In the early days of mathematics education research, theories of mathematics as a discipline were only sporadically found on its agenda, but since the 1980's it is customary for many a publication to declare its own position in this respect.

Traditionally, *statistics and exploratory data* analysis have occupied a prominent position as *the* tools for quantitative research. In the early days of mathematics education research, quantitative studies were predominant in empirical research, and even though different sorts of empirical studies entered the field in the 1980's and rapidly grew in significance and popularity amongst researchers, quantitative – and above all statistical – methods remain important tools.

Evidently, the theories in question are the theories of (applied) statistics (hypothesis testing, analysis of variance, regression analysis), including exploratory data analysis, and psychometrics.

In addition to specific theories about the epistemology of mathematics, general *philosophical theories*, not specifically modified or tailored to dealing with mathematics, primarily theories of knowledge, are often invoked in mathematics education research (e.g. Bachelard, Kant, Peirce, Piaget (as a structuralist), Popper, Wittgenstein).

From the earliest days of mathematics education research, *general psychological theories* have been put to use in the field. This is easy to understand as early researchers took it for granted that the only extra-mathematical discipline of relevance to the study of mathematics education beside statistics was psychology. This is also reflected in the fact that the first international study group in mathematical education affiliated to ICMI was the study group on the psychology of mathematics education, PME, which at times even considered itself as *the* international group for mathematics education research. The psychological theories which have been invoked are behaviourism (e.g. Thorndike, Skinner) and neo-behaviourism, cognitive structuralism (Piaget as a psychologist, and followers), cognitive science in general, activity theory (e.g. Vygotsky, 1978; and partly Krutetskii, 1976), psychoanalysis. We can also find psychological theories specifically tailored or developed to focus on mathematics education (e.g Skemp, 1987; Fischbein, 1987; Vergnaud, 1991; APOS (the piagetian Action-Process-Object-Scheme theory developed by Dubinsky and others (e.g. Czarnocha et al., 1999)).

Also *general pedagogical theories* about teaching and learning considered in psycho-social environments (e.g. concerning gender) are put to use in mathematics education research.

The same is true of theories from *linguistics* (structuralism, Chomsky, linguistic registers, semiotics, socio-linguistics (e.g. Bernstein)).

Today, theories from the *socio-cultural sciences* have gained considerable momentum in mathematics education. This has happened along with the extension of our fields of attention and vision to include classrooms and their sub-groups, institutional contexts, social classes and groups, including cultural or ethnic minorities, socio-economic and political issues, etc. Thus the theories employed are imported from sociology (e.g. Beck, Bourdieu, Giddens, Habermas, Luhmann), anthropology (e.g. Lave, Saxe, Wenger), education (e.g. Dewey), political science and history. One of the main methodologies employed in current mathematics education research, *qualitative studies*, is rooted in these sciences, in particular anthroplology and ethnology.

Finally, within this group of theories, recent trends focus on new developments in *neuroscience* which, by means of new brain scanning techniques, allow for studies of the working brain. This has given rise to physiological theories about the organisation of the brain and the mind that seem to have some bearing on mathematical cognition, especially arithmetical concept formation, and problem solving (e.g. Dehaene, 1997; Lakoff and Nuñez, 1997).

Beside theories such as the ones mentioned, mathematics education research has also developed what is sometimes called “homegrown” theories designed to deal with a few or several aspects of mathematics education. Usually such theories are informed by extraneous theories like the ones listed above, so by using the word "homegrown" we are not implying that the seeds themselves cannot be imported into the field from outside. The important thing is that the seeds are sown in the soil of mathematics education proper and that the resulting plants are cultivated in the discipline's own gardens.

Apart from attempts - by Steiner and others (Steiner, 1985) - going back to the early 1980's to provide a foundation for a comprehensive *"Theory of Mathematics Education"*, most home-grown theories focus on a limited set of aspects. Above we have already mentioned the so-called *APOS theory* (Action-Process-Object-Scheme) which Dubinsky (op. cit.) and colleagues have developed as an extension of Piaget's work. This theory focuses on the learning of mathematics, in particular the formation of mathematical concepts, and is seen by some of those who advocate it as sufficient to cover most of the relevant features of mathematical learning. Tall and Vinner (1981), as well as Sfard (1991), and others, have laid the ground for a theory of mathematical concepts based on a distinction between *concept definition* and *concept image* and a distinction between *process* aspects and *object* aspects of concepts. The French school of *didactique des mathématiques* is based on three theories. Brousseau's theory of *didactical situations* in which the point of attention is teaching and teacher orchestrated student activity in the mathematics classroom (Brousseau, 1997). Chevallard's theory of the *didactical transposition* of scientific mathematical content into curricular mathematics (Chevallard, 1991), and Vergnaud's theory of *conceptual fields*, which focuses on the formation of mathematical concepts – especially in arithmetic – and sense-making thereof (Vergnaud, 1991). Yackel and Cobb (1996) have developed aspects of a theory of what has been called *socio-mathematical norms*, which focuses on the socially defined boundary conditions for what it means to be involved in mathematical activity. Schoenfeld and colleagues (Schoenfeld, 1999) have outlined what is meant to be a theory of *mathematics teaching*, Ball and colleagues (Ball et al., 2001, Hill and Ball, 2004) have identified elements of a theory of *teachers' mathematical knowledge for teaching,* and Skovsmose (1994) has worked towards a theory of *critical mathematical education*, in which he focuses on the social and societal roles of mathematics education and the possibilities of empowering students with a critical attitude to the use and misuse of mathematics in society.

Unfortunately, there is no room in this paper to carry through what is actually badly needed, a thorough and critical examination of the sense and extent to which each of the theories named as such in this section is a theory according to the definition I proposed in the previous section. It would be a significant and worthy task, which would further the scientific and scholarly health of our field, if such analyses were undertaken. Should the result of such analyses be that it is not quite justified to speak about theories, it would still be warranted to speak about theoretical perspectives.

Where do the theories come from?

As appears from the previous section, the far majority of the theories invoked or employed in mathematics education research come from *disciplines outside mathematics education* itself. Historically, statistics and psychology were the first "exporters" of theories into mathematics education research, statistics as the main tool in quantitative research paradigms, and they continue to play significant roles in the field, even though their relative importance has decreased as a consequence of the introduction of other theoretical perspectives.

Theories beyond psychology and statistics were introduced in mathematics education research more or less one by one, along with the adoption of new research issues and foci on the agenda of the field. These issues and foci include curriculum reform; the application of mathematics in extra-mathematical domains; philosophical characteristics of mathematics; mathematics classrooms; gender issues; linguistic issues; socio-cultural issues, including minority issues; student beliefs, affections, career perspectives; teacher preparation, in-service training, and teacher attitudes and beliefs; etc. (Niss, 2004)

When such new horizons are opened, we look to other disciplines that have dealt with similar issues and topics and seek their assistance.. However, when it turns out that the assistance offered by other disciplines is somewhat limited, because they are unable to account for the essential role of *mathematics* in the teaching and learning of mathematics, it becomes necessary to modify, reconstruct, or combine extraneous theories in order to tailor them to fit the needs of mathematics education, and, ultimately, to create specific theories having aspects of mathematics education in focus.

What foundations do the theories have?
On closer inspection, most of the theories which play a role in mathematics education have foundations which are primarily *conceptual* in that they are based on the proposal and introduction of notions, distinctions and concepts of a rather general nature. Sometimes, but not always, the theories have a degree of partial empirical corroboration in the sense that they are mostly inspired by reflection on experience. Only in rare cases do they enjoy the sort of substantiation that may arise from systematic empirical or experimental testing. Only seldom do theories aspire to provide definite claims about the state of affairs in mathematics education. Instead, they are often *interpretive*.

All this implies that the theories we have looked at *never* remain *unchallenged* in the mathematics education research community.

What are their roles and functions?
We saw in an earlier section that theories have different purposes, where the term "purpose" suggests an ultimate end of having a theory at all. The "role" of a theory, which will preoccupy us in this section, is somewhat different in that it points to the actual place of a theory in a larger picture, to the ways in which it is related to other components of the picture, and to the function(s) it serves in that context. With that understanding in mind, different theories have different roles in research in mathematics education.

Some theories serve as an *overarching framework* from which (parts or aspects of) the teaching and learning of mathematics can be viewed and approached. This constitutes a "top-down" approach, where the theoretical framework is given before and outside the specific piece of research in which it is being put to use. In principle – albeit not so much in practice - this implies that the only objects, situations, phenomena, and processes considered are ones that are permitted by and visible from the theoretical framework. Such theories tend to be based on one or a few fundamental concepts, like "reflective abstraction" in Piaget's theory of cognition or in the APOS theory, or like "the zone of proximal development" in Vygotski's activity theory, "didactical" and "adidactical situations" and "the didactical contract" in Brousseau's theory, like "social construction" in social constructivism, and so forth and so on.

Some theories focus on *organising a set of specific observations and interpretations* of singular but related phenomena into a coherent whole. This constitutes a "bottom-up aproach", where no specific theory is imposed on the data – which, then, have not been collected according to a theory-based design – but is supposed to arise from them by concrete analysis. A typical example of this is "grounded theory", which is a way of establishing a specific theory grounded on the data given on the basis of a general methodology. In other words "grounded theory" can actually best be perceived as a meta-theory of how to obtain a specific theory.

Some theories have the role of providing the *terminology* - including the concepts and distinctions that come with it - involved in a particular piece of research. Examples of this are the "process-object duality" of mathematical concepts and "reification" (Sfard, 1991), "procept" (Tall, 1991), "concept image", "epistemological obstacle" (Bachelard, 1938; Sierpinska, 1994), the "epistemological triad" (Steinbring, 1989), "S and I rationales for learning" (Mellin-Olsen, 1981) etc.

Some theories offer a research *methodology*, primarily for empirical studies. Currently, qualitiative methods are prevalent in mathematics education research, and the methodologies involved, which are borrowed from other disciplines in the humanities and the social sciences, help design and analyse observation protocols, interviews, questionnaires, student tasks, or to produce transcripts or video-clips, or to create fictitious individuals representing typical segments of a population, and so on. Meta-theoretical considerations propose method triangulation in empirical research so as to avoid biased interpretations of data caused by a research instrument in itself. When it comes to quantitative studies, applied statistics and its off-springs psychometrics and quantitative sociology continue to prevail as the way to obtain answers to the questions posed.

So far we have concentrated on the roles of theories that are not only mentioned in actual research but in fact utilised. However, when looking at lots of examples of actual research there are numerous cases where a theory is in fact being invoked, but where the relation between the theory and the specific piece of research seems to be missing, i.e. the research is carried out without really involving the theory which is being invoked. This means that references to theory tend to be rhetorical. In such cases, why, then, is a theory being invoked at all? What is its role in the research being conducted?

The only reasonable answer to this question seems to be that since, in such cases, the theory has not informed the research design, process, and inferences, its role must be limited to the publication part of the research. This suggests, then, that a theory is either being invoked in order to legitimise the piece of research at hand, so as to increase its scholarly solidity and credibility, or it serves as a means of announcing the author's adherence to or membership of a particular sub-community of researchers to which it may be seen as desirable to belong. In my view either possibility is unfortunate as it counteracts what ought to be a key feature of everything we do in research: uncompromising honesty about the nature of what we are doing.

Are theories put to use in mathematics education research "good enough"?
The crude and quick answer is "in general, no!". Here are a few details to elaborate on this answer.

Theories imported from other fields are largely *insufficient*, for the following reasons. Firstly, no single imported theory encompassess all of mathematics, mathematics learning and teaching, the relations between all kinds of individuals, groups, classrooms, institutions, communities, and societies with mathematics, nor all significant contexts and dimensions therein and thereof. Secondly, separate theories that deal with different domains pertinent to mathematics education cannot just be glued together so as to form a comprehensive "patchwork" theory of mathematics education. At least, so far nobody has been able to propose such a patchwork and demonstrated its universality. Thirdly, and perhaps most importantly, most imported theories are of a general nature, which does not allow them to offer a sufficient pool of specific results and concrete methodologies so as to provide complete guidelines for conducting a piece of research work.

Theories that are homegrown in mathematics education (and their number is not large) do not suffer from the defiency hinted at in the first of the above-mentioned reasons, lack of specificity to *mathematics* education. However, each of them suffers from lack of comprehensiveness in its endeavour. Similarly, even if forces were combined they too have not been (cannot be?) glued together to form a comprehensive patchwork theory of mathematics education. With regard to the third respect, some homegrown theories seem to have a potential for offering at least partial guidelines for conducting specific pieces of research. For instance, the French school of mathematics education research, combining the theories of Brousseau, Chevallard, Vergnaud, and complemented with Duval's theory of linguistic registers, does seem to outline a paradigm for research in mathematics education, at least in the context of the French educational system.

What should we strive at in theory building in mathematics education?
Imagine that there existed a full-fledged theory of mathematics education. What would it look like, and what *minimum requirements* would it have to fulfil? In order for it to be comprehensive enough to be worth its name, it would have to contain at least the following of subtheories, each accounting for essential traits of mathematics education:

- a sub-theory of *mathematics as a discipline and a subject* in all its dimensions, including is nature and role in society and culture;

- a sub-theory of *individuals' affective notions*, experiences, emotions, attitudes, and perspectives with regard to their actual and potential encounters with mathematics;

- a sub-theory of *individuals' cognitive notions*, beliefs, experiences, and perceptions with regard to their actual and potential encounters with mathematics, and the outcomes thereof;

- a sub-theory of the *teaching of mathematics* seen within all its institutional, societal, national, international, cultural and historical contexts;

- a sub-theory of *teachers of mathematics*, individually and as communities, including their personal and educational backgrounds and professional identities and development.

All sub-theories – and I do not claim that the list just given is exhaustive – have to account for *situating* their objects, situations, phenomena and processes *in all the contexts and environments* that influence them, be they scientific, biological, anthropological, linguistic, philosophical, economic, sociological, political, or ideological. Similarly, they must be geared to deal with both descriptive ("what *is* the case, and why?") and normative ("what *ought to be* the case, and why?") issues.

Moreover, each sub-theory has to live up to the *general requirements of a scientific / scholarly theory* (e.g. as outlined in my definition in the second section of this paper), including accounting for the ways in which its claims are obtained and justified.

The sub-theories *cannot just be juxtaposed*, they have to be *integrated* into a coherent and consistent whole, simply because they deal with issues, problems, and topics of which *mathematics* is a constituent component that lies across them all. In addition, as the sub-theories have non-empty intersections they have to be consistent in the ways they speak about entities and issues on which they overlap.

Conclusion

If we were to engage in the construction of theoretical foundations of mathematics education, it follows from the considerations given above that we – i.e. the didacticians of mathematics - have to be in charge ourselves. Of course, in this endeavour we need prudent import from any other field that has something important to offer. However, even if we need "a little help from our friends", only the mathematics education research community possesses the multifaceted expertise necessary for constructing consistent theoretical frameworks that are rich enough to cover the whole field in its complexity.

This being said, it is unlikely that we shall ever arrive at just one theoretical framework for mathematics education research, unifying all researchers in the field. This is a simple consequence of the fact that it seems possible to create several meaningful sub-theories for each of the five domains we have considered. Already for combinatorial reasons, this suggests that several compehensive, respectable, but competing, theories of mathematics education are likely to arise from our endeavours to establish a theoretical foundation of mathematics education research.

Although these prospects may discourage some from engaging in attempts to establish a theory of mathematics education, I do not think it should. First of all we need to do much more serious work in order to come to grips with the ways in which our field can justifiedly be perceived as the scholarly or scientific discipline we all think it is, or ought to be. Secondly, a most important outcome, even in the short term, of such endeavours would be a much richer and better box of sharp tools for critical analysis of research contributions of whichever kind than are at our disposal for the time being. This, in turn, would help raise the level of reflection and consiousness in our field, and not the least so with respect to novice researchers. One rather immediate result of that would be better research, and that wouldn't be too bad, would it?

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